

Mathematics 100A Quiz 1

Due: Friday 9 October 2020

Instructions: This is a closed note, closed book quiz. Please write clearly and fully explain your solutions. You have 30 minutes from when you begin to read the questions to complete it.

1. This question has two parts.

- (a) Suppose G is a group and H is a subset of G . What must hold for H to be a *subgroup* of G ? In other words, give the definition of a subgroup of a group G .
- (b) Consider the subset S of integers defined as $S = \{\dots, -25, -15, -5, 5, 15, 25, \dots\} = \{10r + 5 : r \in \mathbf{Z}\}$. Is S a subgroup of $(\mathbf{Z}, +)$? If so, explain why. If not, use the definition of subgroups to explain why not.

Proof. For H to be a subgroup, H must

- (a) be closed under the group operation, i.e., $x, y \in H$ implies $xy \in H$
- (b) contain the identity, i.e., $1 \in H$
- (c) be closed under inverses, i.e., $x \in H$ implies $x^{-1} \in H$.

This set S is not a subgroup. For example, it does not contain the identity 0. Nor is it closed under the group operation, as $5+5=10$ but $10 \notin S$. (S is closed under inverses however.) $\square$

2. Let S_3 denote the symmetric group on 3 elements. Write x for the permutation (123) and y for the permutation (12). That is, x takes 1 to 2, 2 to 3 and 3 to 1, while y takes 1 to 2, 2 to 1 and 3 to 3. Recall the relation $yx = x^2y$ that was mentioned in lecture. Consider the element $g = x^{-50}y y y x^4 y x y$ of S_3 . The element g is equal to one of the elements $\{1, x, x^2, y, xy, x^2y\}$; which one is it equal to?

Proof. Because $x^3 = 1$ and $51 = 3 \cdot 17$, one has $x^{-50} = x^{-51}x = x$. Moreover, $y^3 = y$ and $x^4 = x$. Thus $g = xyxyxy$. We can now simplify using the relation $yx = x^2y$ to obtain $yxy = x^2yy = x^2$ so $xyxy = xx^2 = 1$ and thus $g = xy$. $\square$